Comparative Accuracy of Selected Multiple Scattering Approximations

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ABSTRACT

Computational results have been obtained for the plane albedo, total transmission and fractional absorption of plane-parallel atmospheres composed of cloud droplets. These computations, which were obtained using the doubling method, are compared with comparable results obtained using selected radiative transfer approximations. Both the relative and absolute accuracies of asymptotic theory for thick layers and delta-Eddington, Meador–Weaver and Coakley–Chýlek approximations are compared as a function of optical thickness, solar zenith angle and single scattering albedo. Asymptotic theory is found to be accurate to within 5% for all optical thicknesses greater than about 6. On the other hand, the Coakley–Chýlek approximation is accurate to within 5% for thin atmospheres having optical thicknesses less than about 0.2 for most values of the solar zenith angle. Though the accuracies of delta-Eddington and Meador–Weaver approximations are less easily summarized, it can generally be concluded that the delta-Eddington approximation is the most accurate for conservative scattering when the solar zenith angle is small, while the Meador–Weaver approximation is the most accurate for nonconservative scattering ($\omega_0 \leq 0.9$). Selected results from the Eddington approximation are presented to illustrate the effect of delta function scaling in the delta-Eddington approximation. In addition, selected results from the single scattering approximation and asymptotic theory are presented in order to help explain the strengths and limitations of the various approximations.

1. Introduction

In recent years much attention has been devoted to the development of simple and computationally fast analytical approximations to the radiative transfer equation. This has largely been the result of the need to parameterize the radiative properties of clouds and aerosols in general circulation climate models (e. g., see Stephens, 1984). In these and other climate model applications, it becomes necessary to rapidly calculate the plane albedo, total transmission and fractional absorption as a function of optical thickness and solar zenith angle for a wide range of atmospheric conditions.

Among the simplest and most widely used approximations to the radiative transfer equation are the two-stream and Eddington approximations. These approximations have been discussed and analyzed by Irvine (1968), Kawata and Irvine (1970), Shettle and Weinman (1970), Liou (1973, 1974), Coakley and Chýlek (1975), Joseph et al. (1976) and Schaller (1979). Recently, Meador and Weaver (1980) and Zdunkowski et al. (1980) have shown that a whole class of approximate two-stream solutions can be reduced to a standard form with only a few coefficients. These coefficients depend on the solar zenith angle, single scattering albedo and one or more moments of the single scattering phase function, while the general equations for the plane albedo and total transmission depend in addition on the total optical thickness of the layer. In spite of this long history of development, no generally agreed-upon variable ranges exist within which one can use a given approximation with assurance that accurate reflected, transmitted and absorbed flux densities will be obtained.

In comparing results obtained from approximate methods with numerical computations obtained from more exact numerical solutions, it is common practice to present either the plane albedo or total transmission as a function of the cosine of the solar zenith angle (μ_0) for selected values of the total optical depth (τ_t). On some occasions the plane albedo or total transmission are simply tabulated for fixed values of μ_0 and τ_t.

Although climate modeling provides a very real motivation for developing accurate approximations for the reflected and transmitted flux densities over a wide range of atmospheric conditions, remote sensing applications also utilize radiative transfer approximations to interpret experimental results. The delta-Eddington approximation (Joseph et al., 1976), for example, has been used to interpret transmitted solar radiation measurements in the nonconservative terrestrial atmosphere (Carlson and Caverly, 1977) and reflected solar radiation measurements in the nonconservative martian atmosphere (Paige and Ingersoll, 1985), both under the condition that $\tau_t \leq 1$.

Once a layer is sufficiently thick, a diffusion regime is established within the layer which permits the ex-

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tension of the plane albedo, total transmission and fractional absorption to comparable values for a semi-infinite layer (van de Hulst, 1968a, 1980; Sobolev, 1975). In the present study, this asymptotic method for thick layers is compared in accuracy with the single scattering approximation and selected two-stream approximations for $0 \leqslant \mu_0 \leqslant 1$ and $0.1 \leqslant \tau_t \leqslant 100$. Results have been obtained for four values of the single scattering albedo (viz., 1.0, 0.99, 0.9 and 0.8) and for a cloud phase function having an asymmetry factor $g = 0.843$. Following the suggestion of Wiscombe and Joseph (1977), who considered the accuracy of the Eddington approximation for $g \leqslant 0.5$, both absolute and relative errors in the plane albedo, total transmission and fractional absorption are presented. We concentrate our comparisons on asymptotic theory for thick layers and the delta-Eddington, Meador–Weaver (Meador and Weaver, 1980) and Coakley–Chýlek (I) (Coakley and Chýlek, 1975, model 1) approximations for $\omega_0 = 1.0$ and 0.9. Comprehensive results for all four single scattering albedos and for eight different approximations may be found in King and Harshvardhan (1986).

2. Multiple scattering computations

To provide a baseline for assessing the accuracy of various radiative transfer approximations, numerical computations were performed for a model atmosphere composed of cloud droplets. Figure 1 illustrates the phase function employed in these calculations, which is based on Mie theory for a wavelength $\lambda = 0.754$ μm, refractive index $m = 1.332$, and a size distribution of particles of a given radius proportional to $r^6 \exp(-1.6187r)$, where r is the particle radius in μm. This distribution of particles is a gamma distribution with an effective radius of 5.56 μm and an effective variance of 0.111, and is considered typical of fair weather cumulus (FWC) clouds (Hansen, 1971). This distribution is similar to Deirmendjian's (1963) cloud C.1 model, except that the effective radius in Deirmendjian's model is 6.0 μm. The radius range used in our phase function computations was 0.01 to 12.5 μm.

In performing our multiple scattering calculations, we have followed the common practice of expressing the product of the single scattering albedo ω_0 and phase function $\Phi(\cos\Theta)$ as a finite expansion of Legendre polynomials of the form

$$\omega_0\Phi(\cos\Theta) = \sum_{l=0}^{L} \omega_l P_l(\cos\Theta), \qquad (1)$$

where Θ is the scattering angle and $P_l(\cos\Theta)$ a Legendre polynomial of order l. With this definition, the phase function obeys the normalization condition

$$\frac{1}{2}\int_{-1}^{1} \Phi(\cos\Theta)d(\cos\Theta) = 1, \qquad (2)$$

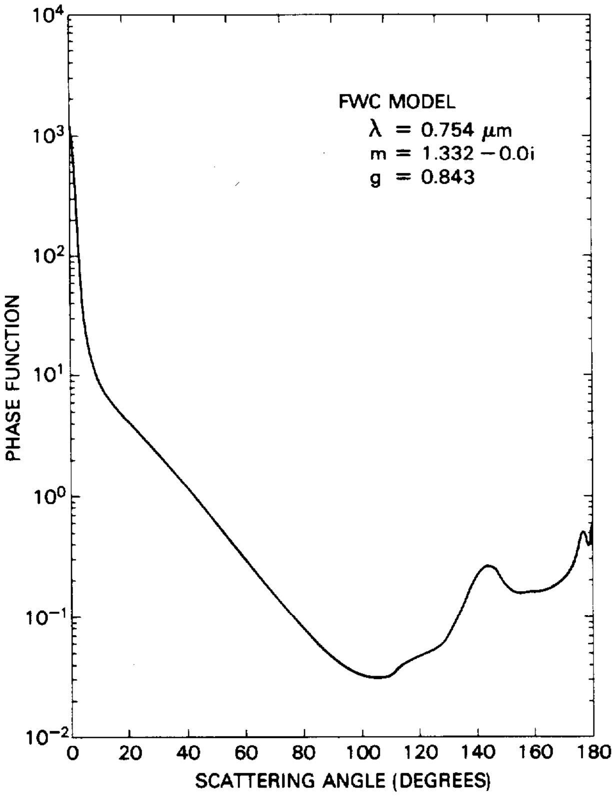

FIG. 1. Phase function as a function of scattering angle for a fair weather cumulus (FWC) size distribution given by $n(r) \propto r^6 \exp(-1.6187r)$, where $\lambda = 0.754$ μm and $m = 1.332 - 0.0i$.

with the asymmetry factor g related to the Legendre coefficient ω_1 by

$$\frac{1}{2}\int_{-1}^{1} \Phi(\cos\Theta)\cos\Theta d(\cos\Theta) = g = \omega_1/(3\omega_0). \qquad (3)$$

The $L = 230$ significant coefficients of the Legendre polynomial expansion of the phase function were evaluated using the orthogonality properties of the Legendre polynomials, together with Gaussian quadrature (see King, 1983, for further details, as well as for an illustration of ω_l as a function of l for a phase function similar to the one illustrated in Fig. 1). The asymmetry factor for this cloud model is $g = 0.843$.

Having determined the Legendre coefficients ω_l, multiple scattering calculations were performed for the azimuth-independent term of the reflection and transmission functions using the doubling method described by Hansen and Travis (1974), together with the invariant imbedding initialization described by King (1983). In terms of these functions, the azimuthally averaged reflected $I^0(0, -\mu)$ and transmitted $I^0(\tau_t, \mu)$ intensities from a horizontally homogeneous atmosphere illuminated from above by a parallel beam of radiation of incident flux density F_0 may be expressed as

$$I^0(0, -\mu) = (\mu_0 F_0/\pi)R^0(\tau_t; \mu, \mu_0), \qquad (4)$$

$$I^0(\tau_t, \mu) = (\mu_0 F_0/\pi) T^0(\tau_t; \mu, \mu_0), \tag{5}$$

where τ_t is the total optical thickness of the atmosphere, μ_0 the cosine of the solar zenith angle and μ the cosine of the zenith angle with respect to the positive τ direction. By convention, the cosine of the zenith angle which appears in the definition of the reflection and transmission functions is defined with respect to the outward-directed normal ($0 \leqslant \mu, \mu_0 \leqslant 1$).

In terms of the azimuth-independent reflection function $R^0(\tau_t; \mu, \mu_0)$ and transmission function $T^0(\tau_t; \mu, \mu_0)$, the plane albedo, total transmission and fractional absorption of the layer are given by

$$r(\tau_t, \mu_0) = 2 \int_0^1 R^0(\tau_t; \mu, \mu_0) \mu d\mu, \tag{6}$$

$$t(\tau_t, \mu_0) = 2 \int_0^1 T^0(\tau_t; \mu, \mu_0) \mu d\mu + \exp(-\tau_t/\mu_0), \tag{7}$$

$$a(\tau_t, \mu_0) = 1 - r(\tau_t, \mu_0) - t(\tau_t, \mu_0). \tag{8}$$

Due to the use of a highly anisotropic phase function and the requirement that accurate computations of the plane albedo, total transmission and fractional absorption be obtained, we have subdivided the μ angular interval using a Gaussian quadrature of order 80 (King, 1983).

Since the major purpose of this study is to examine the accuracy of various radiative transfer approximations over a wide range of optical depths, solar zenith angles and single scattering albedos, we have ignored the effects of surface reflection. As a result, the reflection and transmission functions appearing in (4)–(7) apply to those of an isolated cloud layer only.

Figure 2 illustrates $r(\tau_t, \mu_0) = 1 - t(\tau_t, \mu_0)$ as a function of τ_t and μ_0 for conservative scattering ($\omega_0 = 1$). Figure 3 illustrates doubling computations of the plane albedo $[r(\tau_t, \mu_0)]$, total transmission $[t(\tau_t, \mu_0)]$ and fractional absorption $[a(\tau_t, \mu_0)]$ for nonconservative scattering ($\omega_0 = 0.9$), where we have used the same phase function as illustrated in Fig. 1 but simply scaled the Legendre coefficients by ω_0. The doubling computations used to generate these results were obtained at twelve optical depths 0.0625, 0.125, . . . , 128 and 81 values of the cosine of the solar zenith angle. These μ_0 values include 80 Gaussian quadrature points in addition to the special case $\mu_0 = 1$. From these results we employed a spline under tension interpolation [see Cline (1974) for details] to generate a 300 × 300 matrix of $r(\tau_t, \mu_0)$, $t(\tau_t, \mu_0)$ and $a(\tau_t, \mu_0)$ values that were equally spaced in μ_0 ($0 \leqslant \mu_0 \leqslant 1$) and $\log\tau_t$ ($0.1 \leqslant \tau_t \leqslant 100$). These results, presented in Figs. 2 and 3, represent the exact results to which the radiative transfer approximations are compared in Section 4.

3. Radiative transfer approximations

There are generally three classes of radiative transfer approximations which have been used the most frequently to approximate the plane albedo, total transmission and fractional absorption of a layer. These approximations, which include two-stream approximations, asymptotic theory for thick layers and single scattering, differ considerably in both their assumptions and accuracies as a function of τ_t and μ_0. In the following sections we will outline the assumptions and formulae applicable to each of these methods, as well as explain some of the reasons behind the success and failure of these methods in regimes outside their limits of applicability.

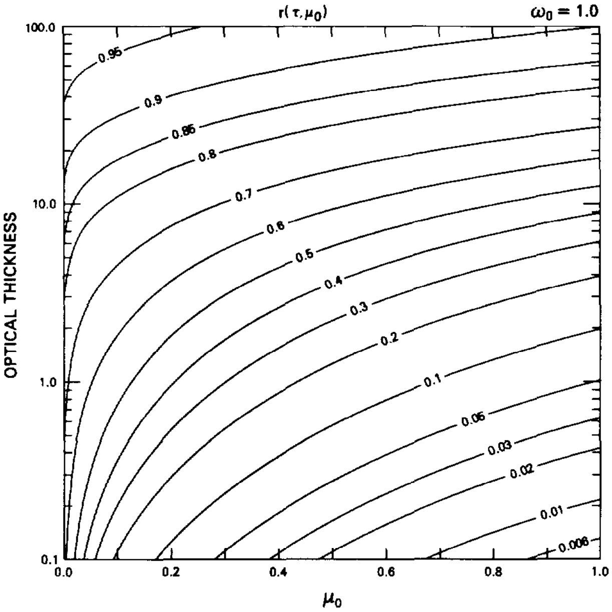

FIG. 2. Computations of the plane albedo as a function of optical thickness and cosine of the solar zenith angle for a FWC phase function with conservative scattering ($\omega_0 = 1.0$).

a. Asymptotic theory

When the optical thickness is sufficiently large, a diffusion regime is established within the layer. Making use of the principles of invariance, together with the requirement that a diffusion domain be established at an optical depth τ_t within a semi-infinite layer, it is straightforward to show that the reflection and transmission functions of a layer of optical thickness τ_t can be expressed in terms of functions applicable to a semi-infinite layer (van de Hulst, 1968a, 1980; Sobolev, 1975). This method, known as asymptotic theory, is a rigorous solution to the equation of transfer in optically thick layers, and as such, makes no assumption about the angular distribution of the intensity field within the medium. From these expressions for the reflection and transmission functions, coupled with the definitions of plane albedo and total transmission given previously, it can be shown that the asymptotic theory approximations for the plane albedo $[\hat{r}(\tau_t, \mu_0)]$, total trans-

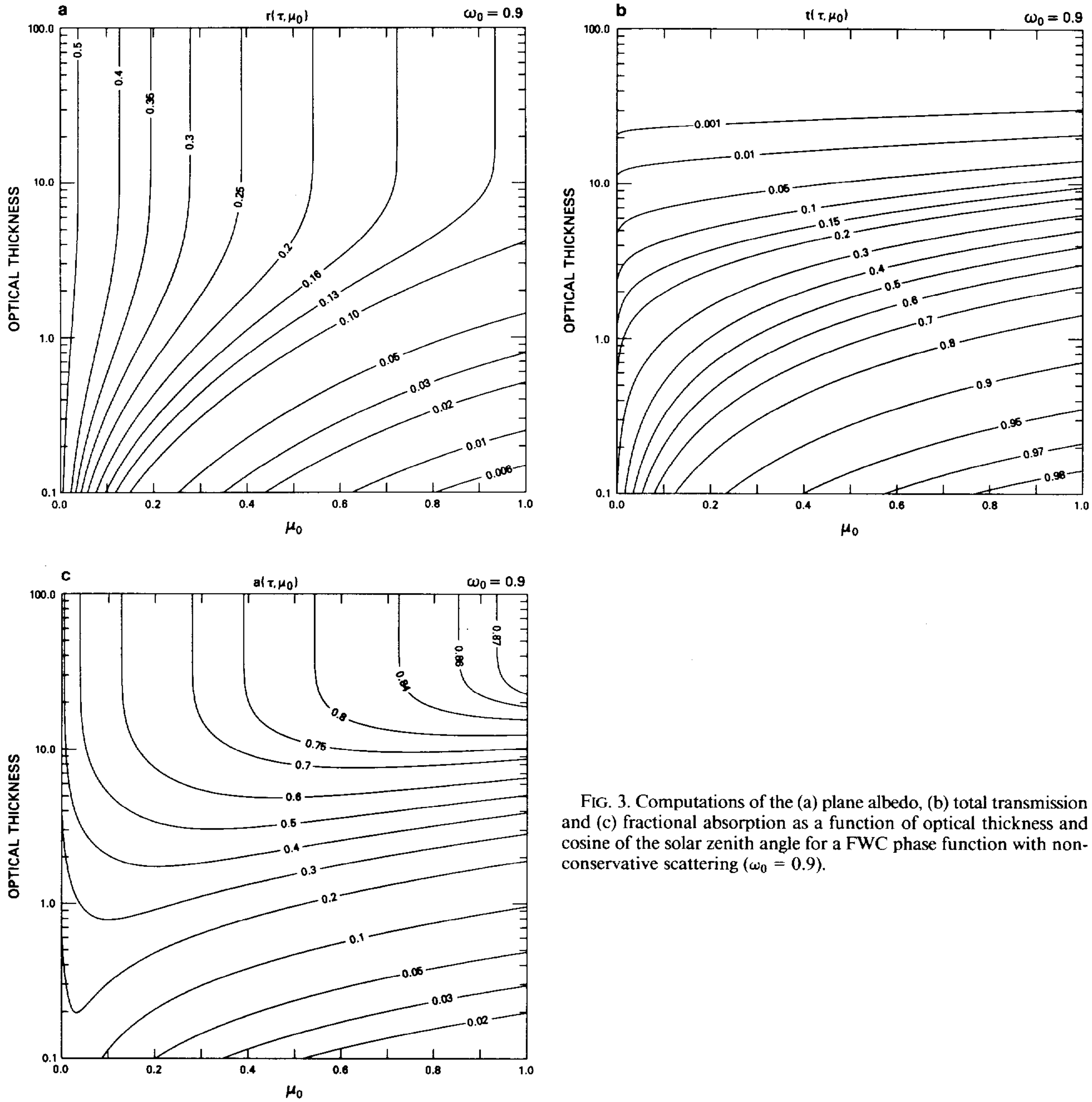

FIG. 3. Computations of the (a) plane albedo, (b) total transmission and (c) fractional absorption as a function of optical thickness and cosine of the solar zenith angle for a FWC phase function with nonconservative scattering ($\omega_0 = 0.9$).

mission $[\hat{t}(\tau_t, \mu_0)]$, and fractional absorption $[\hat{a}(\tau_t, \mu_0)]$ are given by

$$\hat{r}(\tau_t, \mu_0) = r_\infty(\mu_0) - \frac{mnl}{1 - l^2 e^{-2k\tau_t}} K(\mu_0) e^{-2k\tau_t}, \quad (9)$$

$$\hat{t}(\tau_t, \mu_0) = \frac{mn}{1 - l^2 e^{-2k\tau_t}} K(\mu_0) e^{-k\tau_t}, \quad (10)$$

$$\hat{a}(\tau_t, \mu_0) = 1 - \hat{r}(\tau_t, \mu_0) - \hat{t}(\tau_t, \mu_0). \quad (11)$$

In these expressions $r_\infty(\mu_0)$ is the plane albedo of a semi-infinite atmosphere, $K(\mu_0)$ the escape function, k the diffusion exponent describing the attenuation of radiation in the diffusion domain,

$$n = 2 \int_0^1 K(\mu) \mu d\mu, \quad (12)$$

and m and l constants which depend primarily on the single scattering albedo and asymmetry factor (King, 1981). Note that in the derivation of the reflection and transmission functions in asymptotic theory, the role of direct radiation is neglected in comparison with the role of diffuse radiation. As a consequence, (10) could be considered to be either the diffuse or total transmission for collimated radiation, with (11) adjusted accordingly. In examining both of these assumptions, it turns out that (10) is a more accurate representation of the total transmission than the diffuse transmission.

The escape function and diffusion exponent, as well as other asymptotic functions and constants appearing in (9) and (10), can be obtained by applying the asymptotic fitting method of van de Hulst (1968b). In this method, computational results from the doubling method are fit to asymptotic expressions for the plane albedo, diffuse transmission and internal intensity field as a function of optical depth. Figure 4 illustrates the escape function $K(\mu_0)$ as a function of μ_0 for four values of ω_0 and for the FWC phase function illustrated in Fig. 1. It is evident from this figure that the total transmission at the base of an optically thick atmosphere is between 4.35 ($\omega_0 = 1$) and 13.67 ($\omega_0 = 0.8$) times greater for overhead sun ($\mu_0 = 1$) than for grazing incidence ($\mu_0 = 0$). For a given value of ω_0, the escape function is primarily a function of the asymmetry factor, showing little sensitivity to the higher order moments of the phase function (King, 1983).

In addition to the escape function, the plane albedo and total transmission of thick layers depend on the constants m, n, l and k [cf. Eqs. (9) and (10)]. Each of these constants is strongly ω_0 dependent with a somewhat weaker dependence on g. Though one might expect each constant to depend on all expansion coefficients of the phase function, it turns out that the higher order expansion coefficients are quite insignificant.

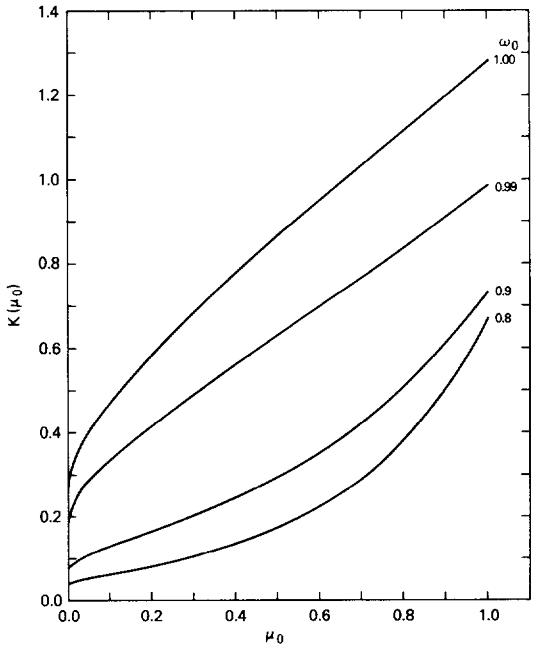

FIG. 4. Escape function as a function of cosine of the solar zenith angle for a FWC phase function and for four values of the single scattering albedo.

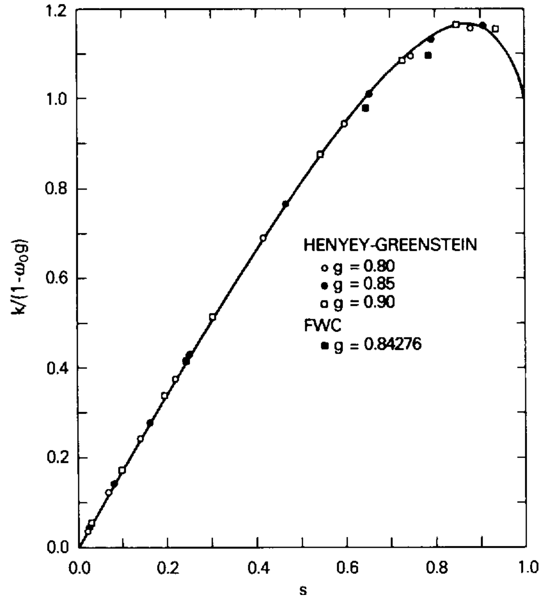

FIG. 5. $k/(1 - \omega_0 g)$ as a function of similarity parameter $s = [(1 - \omega_0)/(1 - \omega_0 g)]^{1/2}$, where k is the diffusion exponent. The symbols represent values obtained by numerical computation for FWC and Henyey–Greenstein phase functions, and the curve the result of a least-squares fit to an analytic equation.

King (1981) examined this question of similarity and found that m, n and l can be well described by a function of a similarity parameter s, defined by

$$s = \left(\frac{1 - \omega_0}{1 - \omega_0 g}\right)^{1/2}, \tag{13}$$

where s reduces to $(1 - \omega_0)^{1/2}$ for isotropic scattering and spans the range 0 ($\omega_0 = 1$) to 1 ($\omega_0 = 0$).

Although the diffusion exponent k does not obey such a similarity relationship, the function $k/(1 - \omega_0 g)$ does. Figure 5 illustrates $k/(1 - \omega_0 g)$ as a function of s for both the FWC ($\omega_0 = 0.99$, 0.9 and 0.8) and Henyey–Greenstein phase functions for varying values of ω_0 (0.9999, 0.999, 0.996, 0.99, 0.96, 0.9, 0.8 and 0.6) and g (0.8, 0.85 and 0.9). The Henyey–Greenstein phase function, first introduced by Henyey and Greenstein (1941), is given by

$$\Phi(\cos\Theta) = \frac{1 - g^2}{(1 + g^2 - 2g\cos\Theta)^{3/2}}. \tag{14}$$

This phase function is often utilized in radiative transfer applications because of the simple expression which results for the Legendre coefficients of the phase function, viz., $\omega_l = (2l + 1)g^l\omega_0$.

The computational results presented in Fig. 5 were fit to an empirical formula in order to give a satisfactory fit to $k/(1 - \omega_0 g)$ as a function of s. The fit thus obtained is presented as a smooth curve in Fig. 5. It has the desirable characteristics that $k = 0(1)$ at $\omega_0 = 1(0)$, and

$$k/(1 - \omega_0 g) = \sqrt{3}s + O(s^2), \tag{15}$$

where $O(s^2)$ denotes terms of the order s^2 or higher. The formulas for m, n, l and $k/(1-\omega_0 g)$ are summarized in Table 1. The formula for m is identical to that obtained by King (1981), whereas the coefficients in the formulas for n and l differ slightly in order to give a better fit for small values of s.

For conservative scattering, when $r_\infty(\mu_0) = n = l = 1$ and $m = k = 0$, the asymptotic expressions for the plane albedo and total transmission given in (9) and (10) are indeterminate. Expanding n, l, m and k to first order in s, it can be shown that (9) and (10) can be rewritten as (King, 1981)

$$\hat{r}(\tau_t, \mu_0) = 1 - \frac{4K(\mu_0)}{3(1-g)(\tau_t + 2q_0)}, \tag{16}$$

$$\hat{t}(\tau_t, \mu_0) = \frac{4K(\mu_0)}{3(1-g)(\tau_t + 2q_0)}, \tag{17}$$

where q_0 is the extrapolation length. The reduced extrapolation length $q' = (1-g)q_0$ is known to range between 0.709 and 0.715 for all possible phase functions (van de Hulst, 1980), and can be well approximated by 0.714 for anisotropic cloud phase functions (King, 1981). Thus it is seen that the plane albedo and total transmission in optically thick, conservatively scattering layers are a function of $(1-g)\tau_t$, with all of the solar zenith angle dependence contained in $K(\mu_0)$.

b. Two-stream approximations

The two-stream approximations in radiative transfer are based on assuming various analytic forms for the upward and downward intensity fields within and at the boundaries of a plane-parallel medium. Substituting the assumed angular distribution into the integro–differential form of the equation of transfer results in a set of differential equations for the upward and downward diffuse flux densities $F^\pm(\tau, \mu_0)$ (Meador and Weaver, 1980; Zdunkowski et al., 1980):

$$\frac{dF^-(\tau, \mu_0)}{d\tau} = \gamma_1 F^-(\tau, \mu_0) - \gamma_2 F^+(\tau, \mu_0) - F_0\omega_0\gamma_3 e^{-\tau/\mu_0}, \tag{18}$$

TABLE 1. Similarity relations satisfied by the constants which arise in asymptotic expressions for the plane albedo, total transmission and fractional absorption of thick layers.

$$l = \frac{(1 - 0.681s)(1 - s)}{(1 + 0.792s)}$$

$$n = \left[\frac{(1 + 0.414s)(1 - s)}{(1 + 1.888s)}\right]^{1/2}$$

$$m = (1 + 1.537s)\ln\left[\frac{1 + 1.800s - 7.087s^2 + 4.740s^3}{(1 - 0.819s)(1 - s)^2}\right]$$

$$k/(1 - \omega_0 g) = \sqrt{3}s - \frac{(0.985 - 0.253s)s^2}{(6.464 - 5.464s)}$$

$$\frac{dF^+(\tau, \mu_0)}{d\tau} = \gamma_2 F^-(\tau, \mu_0) - \gamma_1 F^+(\tau, \mu_0) + F_0\omega_0\gamma_4 e^{-\tau/\mu_0}, \tag{19}$$

where

$$F^\pm(\tau, \mu_0) = 2\pi \int_0^1 I^0(\tau, \pm\mu)\mu d\mu. \tag{20}$$

Due to our choice of positive (negative) μ for downward (upward) directed radiation, $F^-(\tau, \mu_0)$ represents the upward flux density and $F^+(\tau, \mu_0)$ the downward diffuse flux density at optical depth τ. Note that this is opposite the choice of Meador and Weaver (1980).

In order to obtain the forms given in (18) and (19), it is often necessary to approximate the scattering phase function in order to integrate the source function analytically. Expressions for the plane albedo, total transmission and fractional absorption are obtained by solving (18) and (19), subject to the boundary conditions $F^-(\tau_t, \mu_0) = F^+(0, \mu_0) = 0$. The results may be obtained in the form (Meador and Weaver, 1980)

$$\hat{r}(\tau_t, \mu_0) = \frac{\omega_0}{(1 - k^2\mu_0^2)[(k + \gamma_1)e^{k\tau_t} + (k - \gamma_1)e^{-k\tau_t}]} \times [(1 - k\mu_0)(\alpha_2 + k\gamma_3)e^{k\tau_t} - (1 + k\mu_0)(\alpha_2 - k\gamma_3)e^{-k\tau_t} - 2k(\gamma_3 - \alpha_2\mu_0)e^{-\tau_t/\mu_0}], \tag{21}$$

$$\hat{t}(\tau_t, \mu_0) = e^{-\tau_t/\mu_0}\left\{1 - \frac{\omega_0}{(1 - k^2\mu_0^2)[(k + \gamma_1)e^{k\tau_t} + (k - \gamma_1)e^{-k\tau_t}]} \times [(1 + k\mu_0)(\alpha_1 + k\gamma_4)e^{k\tau_t} - (1 - k\mu_0)(\alpha_1 - k\gamma_4)e^{-k\tau_t} - 2k(\gamma_4 + \alpha_1\mu_0)e^{\tau_t/\mu_0}]\right\}, \tag{22}$$

where

$$\alpha_1 = \gamma_1\gamma_4 + \gamma_2\gamma_3, \tag{23}$$

$$\alpha_2 = \gamma_1\gamma_3 + \gamma_2\gamma_4, \tag{24}$$

$$k = (\gamma_1^2 - \gamma_2^2)^{1/2}, \tag{25}$$

$$\gamma_4 = 1 - \gamma_3, \tag{26}$$

and the fractional absorption $\hat{a}(\tau_t, \mu_0)$ is given by (11).

The γ_1, γ_2 and γ_3 coefficients in (19) and (20) for various two-stream approximations, along with references to their original description in the literature, are given in Table 2. Several of these methods employ delta scaling (Joseph et al., 1976) in which a fraction f of the scattered energy is considered to be in the forward peak, approximated as a Dirac-delta function. For each of these methods, which include the delta-Eddington, Practical Improved Flux Method (PIFM) and delta-discrete ordinates methods, (21) and (22) can still be used as long as the following transformations are made in the coefficients and solutions:

$$\tau_t \rightarrow \tau_t' = (1 - \omega_0 f)\tau_t, \tag{27}$$

$$\omega_0 \rightarrow \omega_0' = (1 - f)\omega_0/(1 - \omega_0 f), \tag{28}$$

TABLE 2. Summary of γ_i coefficients in selected two-stream approximations.

Method	Reference	γ_1	γ_2	γ_3
Eddington	Kawata and Irvine (1970)	$1/4[7 - \omega_0(4 + 3g)]$	$-1/4[1 - \omega_0(4 - 3g)]$	$1/4(2 - 3g\mu_0)$
delta-Eddington	Joseph et al. (1976)	$1/4[7 - \omega_0'(4 + 3g')]$	$-1/4[1 - \omega_0'(4 - 3g')]$	$1/4(2 - 3g'\mu_0)$
PIFM	Zdunkowski et al. (1980)	$1/4[8 - \omega_0'(5 + 3g')]$	$3/4[\omega_0'(1 - g')]$	$1/4(2 - 3g'\mu_0)$
discrete ordinates	Liou (1973, 1974)	$\sqrt{3}/2[2 - \omega_0(1 + g)]$	$\sqrt{3}/2[\omega_0(1 - g)]$	$1/2(1 - \sqrt{3}g\mu_0)$
delta-discrete ordinates	Schaller (1979)	$\sqrt{3}/2[2 - \omega_0'(1 + g')]$	$\sqrt{3}/2[\omega_0'(1 - g')]$	$1/2(1 - \sqrt{3}g'\mu_0)$
Coakley–Chýlek (I)	Coakley and Chýlek (1975)	$\{1 - \omega_0[1 - \beta(\mu_0)]\}/\mu_0$	$\omega_0\beta(\mu_0)/\mu_0$	$\beta(\mu_0)$
Coakley–Chýlek (II)	Coakley and Chýlek (1975)	$2[1 - \omega_0(1 - \bar{\beta})]$	$2\omega_0\bar{\beta}$	$\beta(\mu_0)$
Meador–Weaver	Meador and Weaver (1980)	$\dfrac{7 - 3g^2 - \omega_0(4 + 3g) + \omega_0 g^2[4\beta(\mu_0) + 3g]}{4[1 - g^2(1 - \mu_0)]}$	$\dfrac{-1 + g^2 + \omega_0(4 - 3g) + \omega_0 g^2[4\beta(\mu_0) + 3g - 4]}{4[1 - g^2(1 - \mu_0)]}$	$\beta(\mu_0)$

$$g \rightarrow g' = (g - f)/(1 - f). \tag{29}$$

The primed quantities in (27)–(29), when substituted into the expressions for γ_1, γ_2 and γ_3 (cf. Table 2), as well as into (21) and (22), yield the relevant expressions for $\hat{r}(\tau_t, \mu_0)$, $\hat{t}(\tau_t, \mu_0)$ and $\hat{a}(\tau_t, \mu_0)$ for the delta-scaled approximations. Though various choices of f are possible, the most frequently used choice, and the one used in all computational results to be presented below, is $f = g^2$.

When $\tau_t/\mu_0 \ll 1$, it can be shown that (21) and (22) reduce to

$$\hat{r}(\tau_t, \mu_0) = \frac{\omega_0\tau_t}{\mu_0}\gamma_3 = 1 - \hat{t}(\tau_t, \mu_0) - \frac{\tau_t}{\mu_0}(1 - \omega_0), \tag{30}$$

whereas the equation of transfer reduces to

$$\hat{r}(\tau_t, \mu_0) = \frac{\omega_0\tau_t}{\mu_0}\beta(\mu_0) = 1 - \hat{t}(\tau_t, \mu_0)e^{\tau_t/\mu_0} + \frac{\omega_0\tau_t}{\mu_0}. \tag{31}$$

In this expression $\beta(\mu_0)$ is the backscatter fraction defined by

$$\beta(\mu_0) = \frac{1}{2\omega_0}\int_0^1 h^0(-\mu, \mu_0)d\mu, \tag{32}$$

where $h^0(-\mu, \mu_0)$ is the azimuth-independent redistribution function, defined as the azimuthal average of $\omega_0\Phi(\cos\Theta)$ for incident solar radiation in the direction μ_0 and reflected radiation in the direction $-\mu$ [see King (1983) for an illustration of $h^0(-\mu, \mu_0)$ for a phase function nearly identical to the one in Fig. 1].

Using the addition theorem for spherical harmonics, Wiscombe and Grams (1976) have shown that (32) can be rewritten as

$$\beta(\mu_0) = \frac{1}{2} - \frac{1}{4\pi^{1/2}\omega_0}\sum_{l=0}^{L/2}(-1)^l\frac{\Gamma(l + \frac{1}{2})}{\Gamma(l + 2)}\omega_{2l+1}P_{2l+1}(\mu_0). \tag{33}$$

Figure 6 illustrates $\beta(\mu_0)$ as a function of μ_0 for the FWC phase function used in the present investigation. In order for (30) and (31) to be equivalent, it is necessary for $\gamma_3 = \beta(\mu_0)$, which forms the basis for this choice of γ_3 in many of the two-stream methods (cf. Table 2). It is generally believed that (31) is the plane albedo in the limit of single scattering for thin atmospheres, and that approximations that reduce to (31) are accurate in this limit (Coakley and Chýlek, 1975). Since (31) is valid only when $\tau_t/\mu_0 \ll 1$, however, its validity is strongly μ_0 dependent, failing especially in the limit $\mu_0 \rightarrow 0$.

If $\tau_t \ll 1$ but $\mu_0 = 0$, Eq. (21) has a different limiting solution. In this situation it can be shown that $\hat{r}(\tau_t, \mu_0) = \omega_0/2$ for all methods listed in Table 2 except Coakley and Chýlek's model 1, for which we find

$$\hat{r}(\tau_t \ll 1, \mu_0 = 0) = \frac{\omega_0}{2(1 - \omega_0)^{1/2} + 2 - \omega_0}. \tag{34}$$

Since the correct solution for grazing incidence in the limit of single scattering is $\omega_0/2$ (see below), nearly all

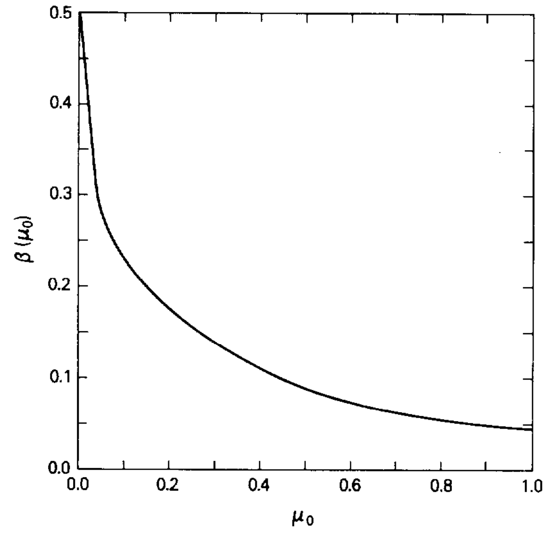

FIG. 6. Backscattering fraction $\beta(\mu_0)$ as a function of cosine of the solar zenith angle.

methods tend to the correct limiting plane albedo for $\mu_0 = 0$ if $\tau_t \ll 1$. Apart from the vicinity of $\mu_0 = 0$, however, (30) governs the behavior of the approximations for thin atmospheres.

Since it is evident that $\gamma_3 > 0$ for physically realistic solutions in the thin atmosphere limit [cf. Eq. (30)], it is expected that the Eddington and discrete ordinates methods will fail this test for a range of μ_0 when the scattering phase function is highly forward-peaked. This failure of the Eddington method is not surprising, since it was developed for isotropic scattering in optically thick, conservatively scattering atmospheres. Delta scaling of the phase function reduces the effective asymmetry factor so that $\gamma_3 > 0$ for all situations, but the condition $\gamma_3 = \beta(\mu_0)$ is met for only a few methods listed in Table 2.

The Coakley and Chýlek (1975) model 2, which we will hereafter refer to as Coakley–Chýlek (II), uses the average backscatter fraction $\bar{\beta}$ in the expressions for γ_1 and γ_2. This constant is defined as

$$\bar{\beta} = \int_0^1 \beta(\mu_0)d\mu_0, \tag{35}$$

and given in terms of the coefficients of the Legendre polynomial expansion of the phase function by (Wiscombe and Grams, 1976)

$$\bar{\beta} = \frac{1}{2} - \frac{1}{8\pi\omega_0}\sum_{l=0}^{L/2}\left[\frac{\Gamma(l+\frac{1}{2})}{\Gamma(l+2)}\right]^2\omega_{2l+1}. \tag{36}$$

For the phase function used in the present investigation, $\bar{\beta} = 0.1772$.

For conservative scattering, for which $\gamma_1 = \gamma_2$ and $k = 0$, Eqs. (21) and (22) reduce to

$$\hat{r}(\tau_t, \mu_0) = 1 - \hat{t}(\tau_t, \mu_0)$$
$$= \frac{1}{1+\gamma_1\tau_t}[\gamma_1\tau_t + (\gamma_3 - \gamma_1\mu_0)(1 - e^{-\tau_t/\mu_0})]. \tag{37}$$

For Coakley and Chýlek's model 1, hereafter referred to as Coakley–Chýlek (I), $\gamma_3 = \gamma_1\mu_0$ and thus (37) reduces to an especially simple form (see Table 2).

The asymptotic behavior of two-stream approximations for semi-infinite atmospheres may be studied using the limiting result for the plane albedo as $\tau_t \to \infty$, derived from (21) and given by (Meador and Weaver, 1980)

$$\hat{r}_\infty(\mu_0) = \frac{\omega_0(\alpha_2 + k\gamma_3)}{(1 + k\mu_0)(k + \gamma_1)}. \tag{38}$$

The plane albedo of a semi-infinite atmosphere is illustrated in Fig. 7 as a function of μ_0 for the FWC phase function and three values of the single scattering albedo. This figure shows the asymptotic behavior of the delta-Eddington (Joseph et al., 1976) and Meador–Weaver (Meador and Weaver, 1980) approximations $[\hat{r}_\infty(\mu_0)]$, as well as the correct asymptotic limit obtained by applying the asymptotic fitting method of van de Hulst (1968b) to doubling computations $[r_\infty(\mu_0)]$. In applying either (37) or (38) to any of the delta-scaled approximations, such as the delta-Eddington approximation, it is of course necessary to make the substitutions given by (27)–(29).

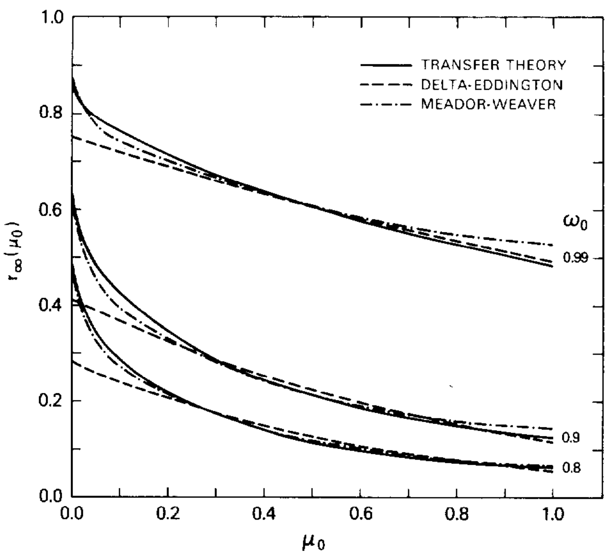

FIG. 7. Plane albedo of a semi-infinite atmosphere $r_\infty(\mu_0)$ as a function of cosine of the solar zenith angle for three values of the single scattering albedo. Accurate radiative transfer results are compared with results from the delta-Eddington and Meador–Weaver approximations.

Figure 7 clearly shows that the Meador–Weaver approximation is able to better simulate the sharp increase in plane albedo for small values of μ_0 than is the delta-Eddington approximation, especially when $\omega_0 = 0.8$. The delta-Eddington approximation, on the other hand, is more nearly similar to the true asymptotic solution for the important range $0.3 \leq \mu_0 \leq 1.0$, but becomes progressively worse as ω_0 decreases from unity. Since the denominator in (38) is positive for all two-stream methods, the necessary condition for positive plane albedos to be obtained in the semi-infinite limit is that

$$\alpha_2 + k\gamma_3 > 0. \tag{39}$$

This condition is failed by some models and for certain combinations of parameters.

In order for a two-stream method to perform well for optically thick atmospheres, it is necessary for $k\tau_t$ (or $k\tau_t'$ for delta-scaled approximations) to be equal to the product of the diffusion exponent and the total optical thickness of the layer, where the diffusion exponent is illustrated in Fig. 5 and parameterized in Table 1. In both Eddington and delta-Eddington methods, for example, this is equivalent to the statement that the diffusion exponent is given by $[3(1-\omega_0)(1-\omega_0 g)]^{1/2}$, which is identical to (15) if one neglects terms of order s^2 or higher. As a consequence, the diffusion exponent is obtained accurately when $s = 0$ ($\omega_0 = 1$), but is overestimated by $\sqrt{3}$ for $s = 1$ ($\omega_0 = 0$). Furthermore, the Eddington and delta-Eddington methods assume that the angular distribution of the intensity field within the layer is linear in μ. This assumption is rigorous only in the diffusion domain of an optically thick, conservatively scattering layer (King, 1981). This helps to explain the greater success of the Eddington and delta-Eddington methods for conservative scattering in thick atmospheres (see below).

Since k exceeds unity for strongly absorbing atmospheres in all two-stream approximations (cf. Table 2), conditions can easily exist for which $\mu_0 = k^{-1}$, especially in the water vapor bands. Though this condition can lead to a numerical singularity in (21) and (22), the singularity is removable, and when $\mu_0 = k^{-1}$ it is rather straightforward to show that (21) and (22) reduce to

$$\hat{r}(\tau_t, \mu_0) = \frac{\omega_0}{2[(1+\gamma_1\mu_0)e^{\tau_t/\mu_0} + (1-\gamma_1\mu_0)e^{-\tau_t/\mu_0}]} \times \{(\alpha_2\mu_0 + \gamma_3)e^{\tau_t/\mu_0} - [(\alpha_2\mu_0+\gamma_3) + 2(\alpha_2\mu_0 - \gamma_3)\tau_t/\mu_0]e^{-\tau_t/\mu_0}\}, \tag{40}$$

$$\hat{t}(\tau_t, \mu_0) = e^{-\tau_t/\mu_0} \times \left\{1 - \frac{\omega_0}{2[(1+\gamma_1\mu_0)e^{\tau_t/\mu_0} + (1-\gamma_1\mu_0)e^{-\tau_t/\mu_0}]} \times \{[(\alpha_1\mu_0 - \gamma_4) - 2(\alpha_1\mu_0+\gamma_4)\tau_t/\mu_0]e^{\tau_t/\mu_0} - (\alpha_1\mu_0 - \gamma_4)e^{-\tau_t/\mu_0}\}\right\}. \tag{41}$$

This case may be avoided by either applying these formulae when $\mu_0 = k^{-1}$, or by displacing μ_0 by a very small increment and applying (21) and (22), as suggested by Zdunkowski et al. (1980).

In computing fluxes for multi-layer systems overlying a reflecting surface, it is also necessary to compute the albedo and transmission of layers for diffuse radiation. For parameterization purposes, it is usual to compute these quantities for an isotropic incident source. Under this situation, the global (spherical) albedo and global transmission can be obtained by integrating the corresponding plane albedo and total transmission solutions as a function of μ_0:

$$\bar{r}(\tau_t) = 2\int_0^1 \hat{r}(\tau_t, \mu_0)\mu_0 d\mu_0, \tag{42}$$

$$\bar{t}(\tau_t) = 2\int_0^1 \hat{t}(\tau_t, \mu_0)\mu_0 d\mu_0. \tag{43}$$

Inspection of (21) and (22), or even the simpler (37) for conservative scattering, shows that a *general* closed form solution for (42) and (43) does not exist. Although these integrations can be carried out numerically for each specific two-stream model, this is not practical for most modeling applications. For those two-stream models for which $\beta(\mu_0)$ does not explicitly appear in any of the γ_i coefficients, it is in principle possible to analytically integrate (21) and (22) to obtain expressions for $\bar{r}(\tau_t)$ and $\bar{t}(\tau_t)$ for specific models. To the best of the authors' knowledge, this has been done only for $\bar{r}(\tau_t)$ in the delta-Eddington (and hence Eddington) approximation (Wiscombe and Warren, 1980).

Coakley and Chýlek (1975) suggest that (42) and (43) may be avoided by using a second set of globally averaged two-stream equations in which the incident isotropic radiation is treated as an upper boundary condition. In general, however, the results obtained by this approach yield different results from those obtained using (42) and (43) for the same model. For example, the spherical albedo of a semi-infinite layer obtained in this manner for the Eddington approximation yields negative values when (Welch and Zdunkowski, 1982)

$$\omega_0 < \frac{1}{4-3g},$$

whereas that obtained by integrating the plane albedo according to (42) yields positive values of the spherical albedo for all values of ω_0 when $g = 0.843$.

c. Single scattering approximation

When the optical thickness of a layer is sufficiently small, the reflection and transmission functions can be expressed in terms of the single scattering phase function. From these expressions, coupled with the definitions of the plane albedo and total transmission given in (6) and (7), it can be shown that the single scattering approximation for the plane albedo and total transmission may be written as

$$\hat{r}(\tau_t, \mu_0) = \frac{1}{2}\sum_{l=0}^{L}(-1)^l\omega_l P_l(\mu_0) \times \int_0^1 \frac{[1 - e^{-\tau_t(1/\mu+1/\mu_0)}]}{\mu + \mu_0} P_l(\mu)\mu d\mu, \quad (44)$$

$$\hat{t}(\tau_t, \mu_0) = \frac{1}{2}\sum_{l=0}^{L}\omega_l P_l(\mu_0)\int_0^1 \frac{[e^{-\tau_t/\mu} - e^{-\tau_t/\mu_0}]}{\mu - \mu_0} \times P_l(\mu)\mu d\mu + e^{-\tau_t/\mu_0}. \quad (45)$$

In order to formally reduce these expressions, it is necessary to define the function $F_l(\tau_t, u)$ such that

$$F_l(\tau_t, u) = \int_1^\infty \frac{v^{-l}}{v - u}[1 - e^{-\tau_t(v-u)}]dv. \quad (46)$$

Making use of this definition, which can be shown to be equivalent to (Chandrasekhar, 1950)

$$F_l(\tau_t, u) = \int_0^{\tau_t} e^{\tau u}E_l(\tau)d\tau, \quad (47)$$

where $E_l(\tau)$ is the exponential integral of order l, it follows that (44) and (45) can be rewritten in the form

$$\hat{r}(\tau_t, \mu_0) = \frac{1}{2\mu_0}\Big\{\omega_0 F_2(\tau_t, -1/\mu_0) - \omega_1\mu_0 F_3(\tau_t, -1/\mu_0) + \frac{1}{2}\omega_2 P_2(\mu_0) \times [3F_4(\tau_t, -1/\mu_0) - F_2(\tau_t, -1/\mu_0)] - \cdots\Big\}, \quad (48)$$

$$\hat{t}(\tau_t, \mu_0) = e^{-\tau_t/\mu_0}\Big\{1 + \frac{1}{2\mu_0}\Big[\omega_0 F_2(\tau_t, 1/\mu_0) + \omega_1\mu_0 F_3(\tau_t, 1/\mu_0) + \frac{1}{2}\omega_2 P_2(\mu_0) \times [3F_4(\tau_t, 1/\mu_0) - F_2(\tau_t, 1/\mu_0)] + \cdots\Big]\Big\}. \quad (49)$$

The extension of these expressions to expressions for the spherical albedo and global diffuse transmission may be found in King et al. (1984).

The importance of defining the $F_l(\tau_t, u)$ functions in problems involving flux transfer in single scattering, plane-parallel atmospheres has apparently been recognized since King (1913). In recent years, however, numerous investigators have made the premature and erroneous assumption that (44) and (45) can be expanded for small values of τ_t and the resulting expressions integrated, thereby resulting in the often-quoted result given in (31). This neglects the fact that the limits of integration in (44) and (45) apply for all μ in the interval [0, 1], thereby negating the assumption that $\tau_t/\mu \ll 1$ for some portion of the μ integration interval.

Though (48) and (49) are practical for calculations of the plane albedo and total transmission only in situations for which the scattering phase function can be expressed as a low order expansion in Legendre polynomials (i. e., $L \leqslant 2$), these expressions permit the examination of a number of useful cases. For example, if one makes use of power series expansions of the $F_l(\tau_t, u)$ functions, such as those found in van de Hulst (1980), it is relatively straightforward to show that the single scattering expressions for the plane albedo and total transmission reduce to

$$\hat{r}(\tau_t, \mu_0) = \frac{\omega_0\tau_t}{\mu_0}\beta(\mu_0) + \epsilon_r(\tau_t, \mu_0), \quad (50)$$

$$\hat{t}(\tau_t, \mu_0) = e^{-\tau_t/\mu_0}\Big\{1 + \frac{\omega_0\tau_t}{\mu_0}[1 - \beta(\mu_0)] + \epsilon_t(\tau_t, \mu_0)\Big\}, \quad (51)$$

where the error terms in the plane albedo $[\epsilon_r(\tau_t, \mu_0)]$ and total transmission $[\epsilon_t(\tau_t, \mu_0)]$ are negligible only when $\tau_t \ln\tau_t \ll 1$ *and* $\tau_t/\mu_0 \ll 1$.

When $\mu_0 = 0$, it can be shown that (44) reduces to

$$\hat{r}(\tau_t, \mu_0 = 0) = \frac{\omega_0}{2}, \quad (52)$$

regardless of optical depth. In addition, the total transmission reduces to $\omega_0/2$ when both $\mu_0 = 0$ *and* $\tau_t \ln\tau_t \ll 1$. Although all two-stream models listed in Table 2 satisfy this criterion when $\tau_t \ll 1$, with the exception of Coakley–Chýlek's (1975) model 1 [cf. Eq. (34)], it turns out that this subtle restriction on the Coakley–Chýlek (I) method is of no practical importance. This is a consequence of the fact that as $\mu_0 \to 0$, multiple scattering is sufficiently important that (52) accurately approximates the plane albedo to within 5% only when $\tau_t \leqslant 0.004$.

4. Results

We have examined both the absolute and relative accuracies of the plane albedo, total transmission and fractional absorption (where applicable) as a function of τ_t and μ_0 for four values of the single scattering albedo (ω_0 = 1.0, 0.99, 0.9 and 0.8) and for all radiative transfer approximations discussed in the previous section. Figure 8 illustrates a 4 × 3 plot composite of results for conservative scattering and for four of these models, where the first row applies to asymptotic theory and succeeding rows to the delta-Eddington, Meador–Weaver and Coakley–Chýlek (I) approximations. Individual plots in the first column of Fig. 8 represent absolute errors in the plane albedo, defined as

$$\Delta r(\tau_t, \mu_0) = \hat{r}(\tau_t, \mu_0) - r(\tau_t, \mu_0). \quad (53)$$

With this definition, positive (negative) errors indicate that the radiative transfer approximation overestimates (underestimates) the exact albedo, taken as the com-

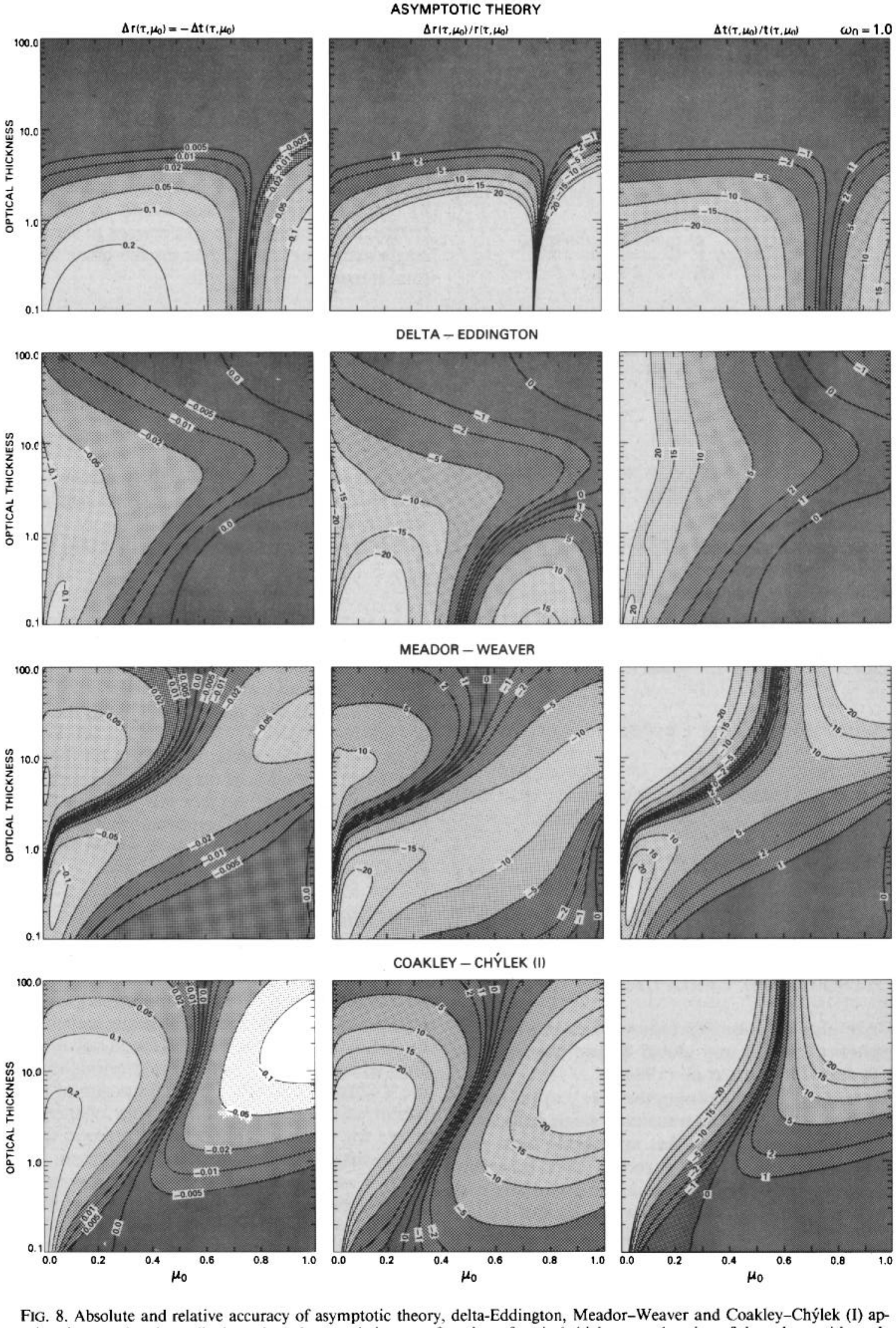

FIG. 8. Absolute and relative accuracy of asymptotic theory, delta-Eddington, Meador–Weaver and Coakley–Chýlek (I) approximations to the plane albedo and total transmission as a function of optical thickness and cosine of the solar zenith angle for conservative scattering ($\omega_0 = 1.0$). All relative accuracy values are in percent. The FWC phase function is assumed throughout.

putational results presented in Fig. 2. Similar definitions apply to errors in the total transmission [$\Delta t(\tau_t, \mu_0)$] and fractional absorption [$\Delta a(\tau_t, \mu_0)$]. The relative errors in the plane albedo [$\Delta r(\tau_t, \mu_0)/r(\tau_t, \mu_0)$] and total transmission [$\Delta t(\tau_t, \mu_0)/t(\tau_t, \mu_0)$] are presented in succeeding columns of Fig. 8, and are given in percent. Relative errors with magnitudes greater than 20% and absolute errors with magnitudes greater than 0.2 are not plotted in Fig. 8 or in subsequent figures.

The approximate values of the albedo, transmission and absorption used to generate these and subsequent graphical results were obtained for the same 300 values of τ_t for which the exact computations were interpolated (cf. section 2), but at the 81 μ_0 values used to perform the doubling computations. These arrays were then interpolated in μ_0 to provide a 300 × 300 matrix of $\hat{r}(\tau_t, \mu_0)$, $\hat{t}(\tau_t, \mu_0)$ and $\hat{a}(\tau_t, \mu_0)$ values, prior to computing errors.

Individual contour plots in Fig. 8 are shaded in order to draw attention to those regions of greatest accuracy. For example, asymptotic theory is seen to be accurate to within 5% in both reflection and transmission for $\tau_t \gtrsim 3$ when $\mu_0 \leqslant 0.9$. Furthermore, asymptotic theory is accurate to within 1% for all solar zenith angles when $\tau_t \gtrsim 8$. The Coakley–Chýlek (I) approximation, on the other hand, is accurate to within 5% for all $\tau_t \leqslant 0.2$ when $\mu_0 \gtrsim 0.1$. Since $r(\tau_t, \mu_0)$ is small when $\tau_t/\mu_0 \leqslant 0.5$ (cf. Fig. 2), a relative error of 5% is too stringent a criterion to use for accepting a model in this range. Adopting instead the absolute error criterion $|\Delta r(\tau_t, \mu_0)| \leqslant 0.005$, we see that the range of validity of the Coakley–Chýlek (I) approximation can be extended to $\tau_t \leqslant 0.5$ for a wide range of solar zenith angles. An advantage of both of these models is that their absolute and relative accuracies show little sensitivity to μ_0 in their respective ranges of validity.

In contrast to these models, the delta-Eddington approximation tends to have its greatest accuracy when $\mu_0 \gtrsim 0.5$, regardless of optical thickness. The large relative albedo errors which occur when τ_t/μ_0 is small are not critical, since the absolute errors are small in this range. Similarly, when $\tau_t \gtrsim 10$ and $\mu_0 \leqslant 0.5$, the small values of $t(\tau_t, \mu_0)$ allow one to extend the range of validity of the delta-Eddington model to values of μ_0 somewhat lower than 0.5. On examination of Table 2, one can readily show that the PIFM method of Zdunkowski et al. (1980) is identical to the delta-Eddington method for conservative scattering, and hence results for the PIFM method are not shown separately. The difference Zdunkowski et al. report between the delta-Eddington and PIFM methods when $\omega_0 = 1$ is likely a result of their using different values of f in the scaling formulae (27)–(29) for each method.

The Meador–Weaver approximation, which was developed as a composite of the Eddington and Coakley–Chýlek (I) methods, has most of the characteristics of the latter for conservative scattering, especially for thick atmospheres. In optically thin atmospheres, on the other hand, it has a much greater μ_0 sensitivity than the Coakley–Chýlek (I) method. This makes the Meador–Weaver approximation less suitable than alternative methods for conservative scattering over the entire range of variables, at least for the high values of asymmetry factor considered in the present investigation.

Detailed results for conservative scattering analogous to Fig. 8 are presented in King and Harshvardhan (1986) for four other radiative transfer approximations, viz., the Eddington, Coakley–Chýlek (II), PIFM and delta-discrete ordinates methods. In addition, contour plots of the approximate albedos $\hat{r}(\tau_t, \mu_0)$ are presented for all eight models. In general, it can be concluded that when $\omega_0 = 1$ the majority of models are inaccurate in their approximation of the plane albedo when $\mu_0 \ll 1$ and $1 \leqslant \tau_t \leqslant 5$, with asymptotic theory being the best suited in this difficult range of variables.

Figure 9 illustrates a 4 × 3 plot composite showing absolute errors in the plane albedo, total transmission and fractional absorption as a function of τ_t and μ_0 for $\omega_0 = 0.9$ and for each of the four models presented in Fig. 8. Corresponding results for relative errors are presented in Fig. 10. In both of these figures, individual plots in the first column represent errors in the plane albedo, while plots in succeeding columns represent errors in the total transmission and fractional absorption, respectively.

Figures 9 and 10 show that asymptotic theory is equally as valid an approximation for optically thick, nonconservative atmospheres as it is for optically thick, conservative atmospheres (cf. Fig. 8). Relative errors of 5% or less are achieved in asymptotic theory for reflection, transmission and absorption when $\tau_t \gtrsim 6$, regardless of solar zenith angle. For cases in which reflection is the most important, the results presented in Fig. 10 show that asymptotic theory can be applied to optical depths as low as 4 with an accuracy of better than 5%.

As in the case of conservative scattering, the Coakley–Chýlek (I) method is the most accurate approximation for optically thin atmospheres, with a tendency to be somewhat more accurate for small solar zenith angles (large values of μ_0). In order to have an accuracy of better than 5% in reflection, transmission and absorption, Fig. 10 suggests that it is necessary for $\tau_t \leqslant 0.1$ and $\mu_0 \gtrsim 0.1$. However, since $a(\tau_t, \mu_0) \ll 1$ when $\tau_t/\mu_0 \leqslant 0.5$ (cf. Fig. 3c), it is more meaningful to use the absolute error criterion $|\Delta a(\tau_t, \mu_0)| \leqslant 0.005$. Thus we conclude that the range of validity of the Coakley–Chýlek (I) approximation can generally be extended to include all optical depths less than some maximum in the range $0.2 \leqslant \tau_t \leqslant 0.7$, depending on solar zenith angle.

For the delta-Eddington approximation, comparison of Figs. 8 and 10 shows that relative errors in the plane albedo and total transmission degrade somewhat in accuracy as absorption increases, especially for optically

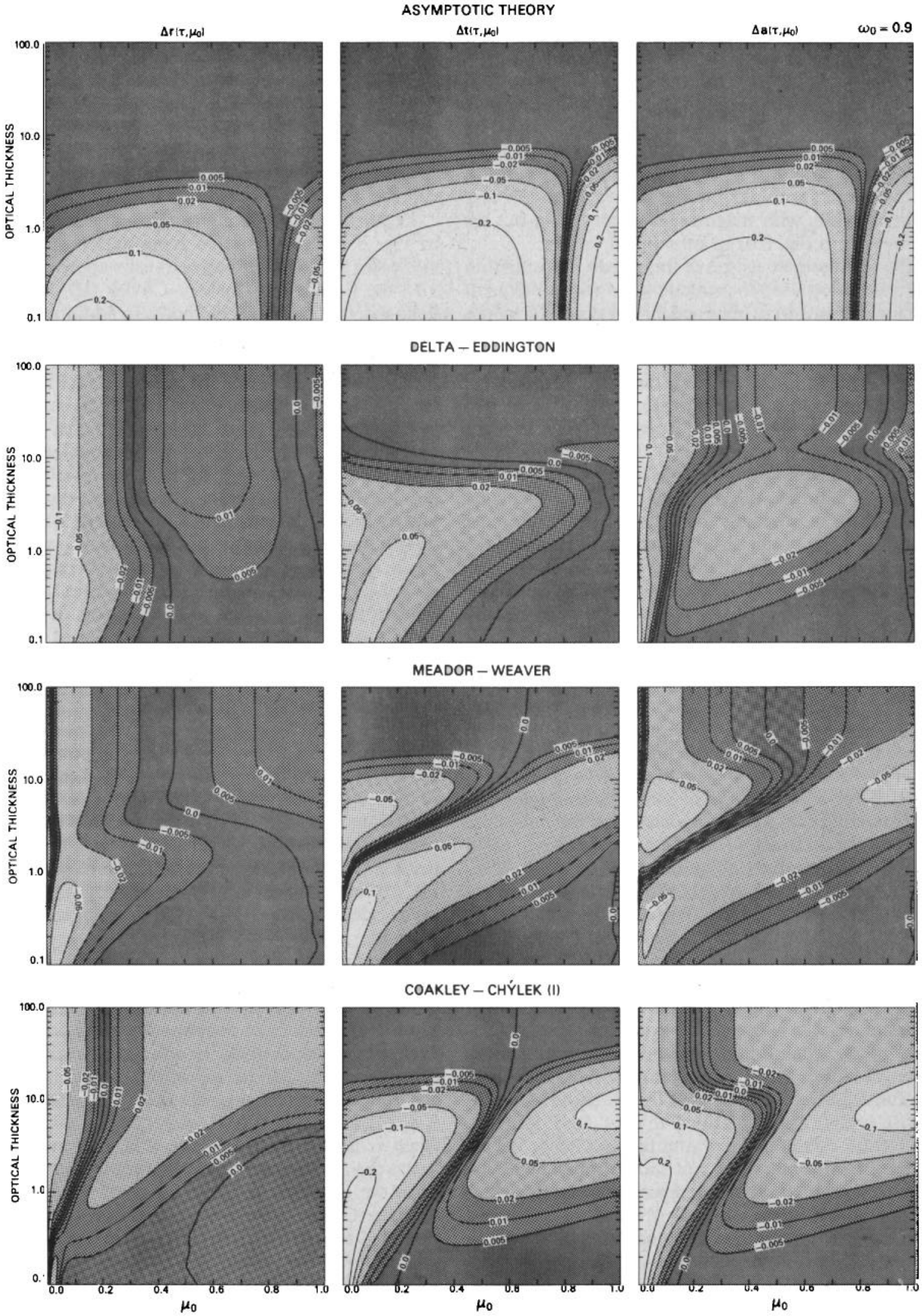

FIG. 9. Absolute accuracy of asymptotic theory, delta-Eddington, Meador–Weaver and Coakley–Chýlek (I) approximations to the plane albedo, total transmission and fractional absorption as a function of optical thickness and cosine of the solar zenith angle for nonconservative scattering ($\omega_0 = 0.9$). The FWC phase function is assumed throughout.

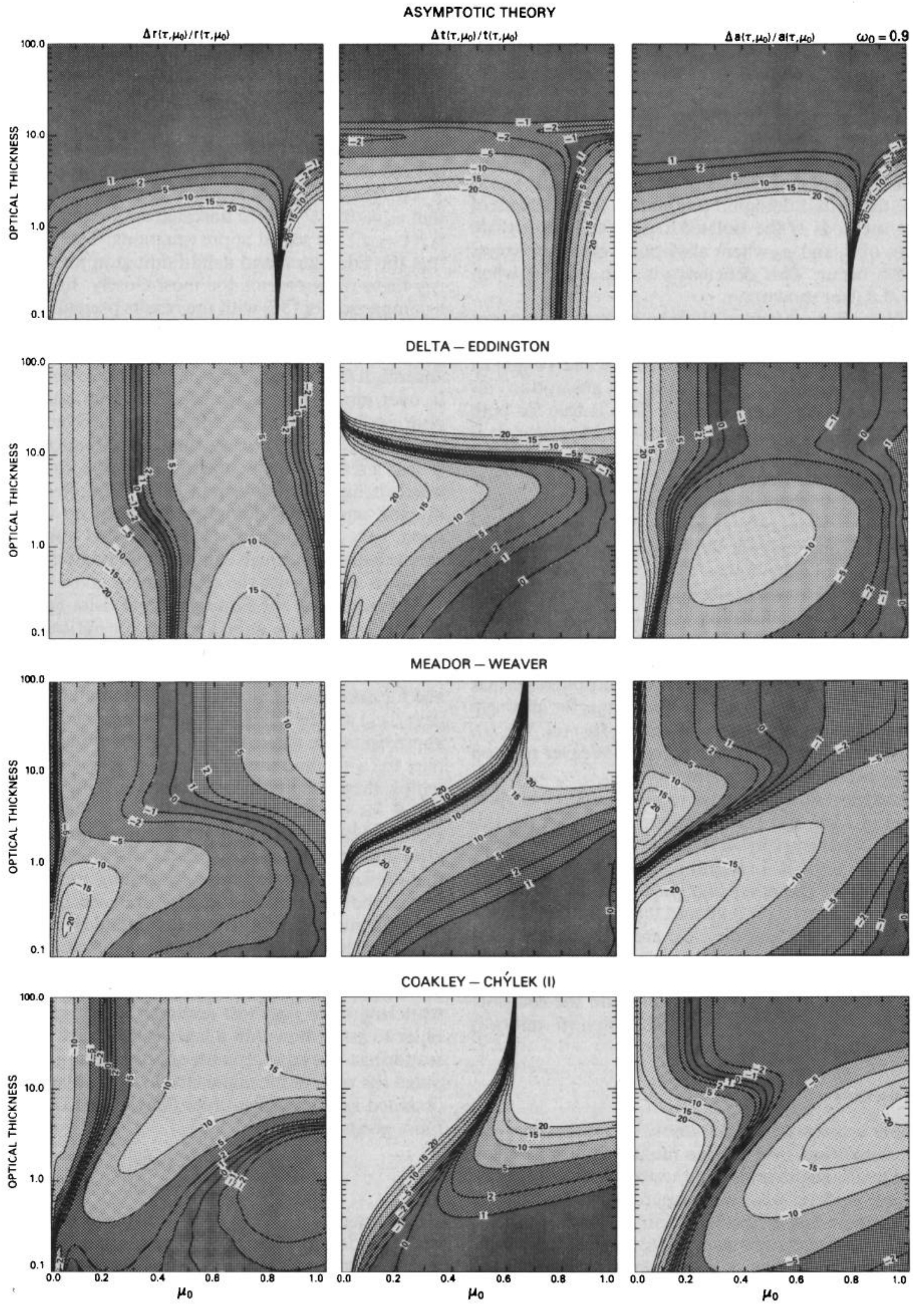

FIG. 10. As is Fig. 9 except for relative accuracies (in percent).

thick atmospheres. This is a consequence of the fact that the $\hat{r}_\infty(\mu_0)$ computed in the delta-Eddington approximation is nearly linear in μ_0 for all single scattering albedos, whereas the true $r_\infty(\mu_0)$ has increasing curvature as absorption increases (cf. Fig. 7). Both the absolute and relative errors in the other delta-scaled approximations (viz., the PIFM and delta-discrete ordinates methods) are very similar to, but slightly worse than, the delta-Eddington method. A feature of all of these methods is the isolated region at intermediate values of τ_t and μ_0 where absorption errors in excess of 10% occur. This deficiency is also present when $\omega_0 = 0.8$ (not shown).

Although the Meador–Weaver approximation was previously shown to be an inferior model for conservative scattering, it is clear from Figs. 9 and 10 that its accuracy improves dramatically as absorption increases, especially for reflection. This is true for both optically thin and thick atmospheres. Moreover, it is the only two-stream model which has an albedo accuracy of better than 5% over a wide range of solar zenith angles when $\tau_t \gtrsim 2$, although the absorption error is sometimes as large as 10% in this range of variables. This tendency for the Meador–Weaver method to improve in accuracy as ω_0 decreases continues at least to $\omega_0 = 0.8$, where albedo errors of less than 7.5% occur for all optical depths when $\mu_0 \gtrsim 0.2$ (King and Harshvardhan, 1986). The explanation for the exceptional accuracy of the Meador–Weaver approximation in optically thick, nonconservative atmospheres is that $\hat{r}_\infty(\mu_0)$ exhibits significant curvature in μ_0 for all single scattering albedos, as does the true $r_\infty(\mu_0)$ (cf. Fig. 7). This feature is unique to the Meador–Weaver method among two-stream approximations.

Detailed results analogous to Figs. 9 and 10 are presented in King and Harshvardhan (1986) for the Eddington, Coakley–Chýlek (II), PIFM and delta-discrete ordinates methods. In addition, error contour plots are presented for the other two single scattering albedos not shown here ($\omega_0 = 0.99$ and 0.8). Contour plots of $\hat{r}(\tau_t, \mu_0)$, $\hat{t}(\tau_t, \mu_0)$ and $\hat{a}(\tau_t, \mu_0)$ are also presented for all eight models and for all single scattering albedos. In general, it can be concluded that all models with the exception of asymptotic theory and the Meador–Weaver approximation have difficulty in optically thick, nonconservative atmospheres.

5. Discussion

After examining a wide variety of radiative transfer approximations over a large range of optical depths, solar zenith angles and single scattering albedos, it has become evident why some approximations succeed while others fail in specific regimes. For example, a straightforward comparison of (16) and (37) shows that when $\omega_0 = 1$ and $\tau_t/\mu_0 \gg 1$, the plane albedo obtained from asymptotic theory and two-stream approximations are equivalent, provided the two-stream coefficients γ_1 and γ_3 satisfy the following criteria:

$$\gamma_1 = \frac{1}{2q'}(1 - g), \tag{54}$$

$$\gamma_3 = 1 + \gamma_1\mu_0 - \frac{2K(\mu_0)}{3q'}. \tag{55}$$

For the delta-scaled approximations, the only difference in these criteria is the substitution $g \rightarrow g'$ in (54). Since $q' \approx 0.714$ for all possible phase functions, (54) implies that $\gamma_1 \approx 0.7(1 - g)$ for unscaled approximations and $0.7(1 - g')$ for scaled approximations. Table 2 shows that the Eddington and delta-Eddington methods satisfy these requirements the most closely. In addition, a comparison of (55) with the results presented in Fig. 4 indicates that in order for a two-stream method to perform well for optically thick, conservative atmospheres, it is necessary for γ_3 to be a linear function of μ_0 over most of the range of solar zenith angles. The poor performance of the Meador–Weaver and Coakley–Chýlek methods under these conditions is at least in part a result of their choice of $\gamma_3 = \beta(\mu_0)$, a function which is highly nonlinear in μ_0 (cf. Fig. 6). The Eddington and delta-Eddington methods, on the other hand, very nearly satisfy (55). Note that although γ_3 can be negative for high sun in the Eddington method, this is in accord with the requirement given by (55).

Although both the Eddington and delta-Eddington approximations are quite accurate for optically thick, conservative atmospheres, they differ substantially in their accuracy at intermediate and small optical depths and for nonconservative scattering. Figure 11 illustrates $\Delta r(\tau_t, \mu_0)$ as a function of τ_t and μ_0 for the Eddington approximation, where the left portion of the figure applies to $\omega_0 = 1$ and the right portion to $\omega_0 = 0.9$. Comparing these results to comparable results in Figs. 8 and 9 for the delta-Eddington approximation shows that the Eddington and delta-Eddington methods are virtually identical for $\omega_0 = 1$ and $\tau_t \gtrsim 15$. Furthermore, these results indicate that delta-scaling not only improves the accuracy of radiative transfer approximations for small optical depths, as intended, but it also improves the accuracy for optically thick atmospheres when $\omega_0 < 1$.

For optically thin atmospheres one expects single scattering to be the most accurate approximation. In order to assess how thin a layer is necessary for single scattering to be an acceptable approximation, we computed the ratio of the plane albedo to the plane albedo obtained assuming first-order (single) scattering. This ratio, given by

$$x(\tau_t, \mu_0) = \frac{r(\tau_t, \mu_0)}{\hat{r}(\tau_t, \mu_0)}, \tag{56}$$

is illustrated in Fig. 12 as a function of τ_t and μ_0, where again the left portion of the figure applies to $\omega_0 = 1$ and the right portion to $\omega_0 = 0.9$. It is evident from Fig. 12 that the enhancement of the first-order plane albedo by multiple scattering is 30% or greater at $\tau_t = 0.1$, regardless of single scattering albedo. As the op-

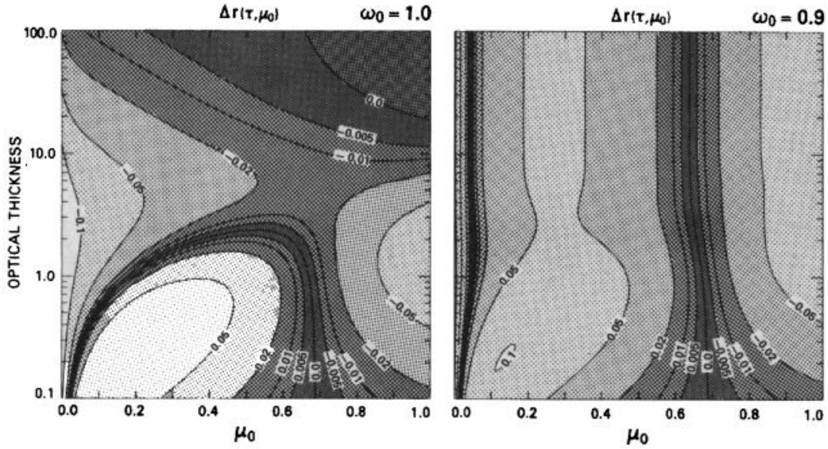

FIG. 11. Absolute accuracy of the Eddington approximation to the plane albedo as a function of optical thickness and cosine of the solar zenith angle. The figure on the left applies to conservative scattering ($\omega_0 = 1.0$) and the figure on the right to nonconservative scattering ($\omega_0 = 0.9$).

tical thickness increases, the multiple scattering enhancement factor $x(\tau_t, \mu_0)$ continues to increase, being everywhere larger for conservative scattering than for nonconservative scattering. This is especially evident for optically thick atmospheres, where multiple scattering enhances the plane albedo by a factor of about 50 (8.4) for $\omega_0 = 1$ (0.9). These results clearly show that single scattering is an inaccurate approximation for all optical depths $\tau_t \gtrsim 0.1$. Some two-stream approximations that tend to the single scattering limit for $\tau_t \ll 0.1$ perform much better than the single scattering approximation when $\tau_t \approx 0.1$. For example, Figs. 8 and 10 show that two-stream approximations seldom have albedo errors as large as 30% when $\tau_t \approx 0.1$. It is therefore always preferable to use appropriate two-stream approximations or asymptotic theory, rather than single scattering, when computing fluxes in atmospheric layers of optical thickness $\tau_t \gtrsim 0.1$.

6. Summary and conclusions

In the present study the plane albedo, total transmission and fractional absorption predicted by various radiative transfer approximations have been compared with doubling computations as a function of optical thickness, solar zenith angle and single scattering albedo. In order to gain insight into the strengths and weaknesses of various two-stream approximations, we studied the limiting behavior of these methods for both optically thick and optically thin atmospheres. This was done by comparing their accuracies with asymptotic theory for thick atmospheres and the single scattering approximation for thin atmospheres.

The results presented in Section 4 have shown that specific regions can be identified where one approximation is more accurate than another. For remote sensing applications involving flux measurements of

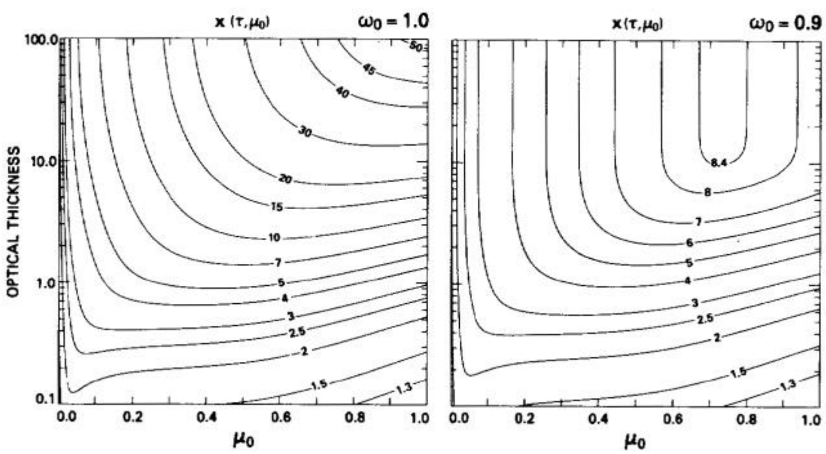

FIG. 12. As is Fig. 11 except for the ratio of the plane albedo to the plane albedo obtained assuming first-order (single) scattering.

either reflected or transmitted radiation, it is generally possible to use these results as a guide in selecting the most accurate approximation to use. For example, the most accurate radiative transfer approximation to use in analyzing diffuse-direct ratio measurements would be the Coakley–Chýlek (I) method, since this method for inferring the single scattering albedo of tropospheric aerosol particles requires modeling the transmission as a function of solar zenith angle (King and Herman, 1979). In addition, the Coakley–Chýlek (I) approximation is the most appropriate method to use in climate investigations involving the reflection and transmission properties of nonabsorbing stratospheric aerosol layers. For optically thick clouds, on the other hand, asymptotic theory is the most appropriate choice for both conservative and nonconservative atmospheres. The Meador–Weaver approximation is the best suited model for investigations involving moderate and strong absorption, such as energy budget studies within water vapor bands or strongly absorbing planetary atmospheres. Arctic haze investigations are the most difficult applications to model accurately using approximate radiative transfer methods, since the combined conditions of low sun, moderate absorption and small optical depths are difficult to meet with any approximation.

For climate model applications in which a single radiative transfer approximation is required to accurately model reflection, transmission and absorption at all optical depths, solar zenith angles and single scattering albedos, it is clear from the results presented in Section 4 that this criterion is not satisfied by any of the approximate methods. In general circulation models in which it is the most important to have accurate computations in moderately thick cloud layers and over a wide range of solar zenith angles, our results show that the delta-Eddington method is better suited than alternative two-stream methods. It must be realized, however, that the treatment of absorption in strong water vapor bands and in optically thin cirrus clouds will be somewhat inaccurate in this method.

Most previous intercomparisons of radiative transfer approximations have concentrated on presenting results for the plane albedo as a function of cosine of the solar zenith angle for selected values of the optical depth. On some occasions, intercomparisons have been further restricted to selected values of μ_0. Although the delta-Eddington approximation is accurate for reflection to within 5% for all optical depths when $\omega_0 = 0.9$ and $\mu_0 = 0.4$ (cf. Fig. 10), a generalized conclusion on its overall accuracy based on this restricted intercomparison would be highly misleading. We therefore feel it is important to examine the accuracy of the plane albedo, total transmission and fractional absorption as a function of optical depth *and* solar zenith angle before drawing conclusions about the overall accuracy of a given approximation.

Due to the importance of developing a multiple scattering approximation which is accurate for all solar zenith angles and over a wide range of optical depths, our results suggest that a hybrid two-stream model that reduces to asymptotic theory for thick atmospheres but extends the range of validity of asymptotic theory to thinner atmospheres would be extremely valuable. The development of such a model remains a challenge for further study. Finally, we would like to note that none of the conclusions drawn in the present investigation is affected by our choice of a Mie theory phase function. Limited intercomparisons with doubling computations using the Henyey–Greenstein phase function with the same asymmetry factor as in the fair weather cumulus model ($g = 0.843$) yield error plots with virtually the same appearance as those in Figs. 8–10.

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REFERENCES

Carlson, T. N., and R. S. Caverly, 1977: Radiative characteristics of Saharan dust at solar wavelengths. *J. Geophys. Res.*, **82**, 3141–3152.

Chandrasekhar, S., 1950: *Radiative Transfer*. Oxford University Press, 393 pp.

Cline, A. K., 1974: Scalar- and planar-valued curve fitting using splines under tension. *Comm. Assoc. Comput. Mach.*, **17**, 218–220.

Coakley, J. A., Jr., and P. Chýlek, 1975: The two-stream approximation in radiative transfer: Including the angle of the incident radiation. *J. Atmos. Sci.*, **32**, 409–418.

Deirmendjian, D., 1963: Scattering and polarization properties of polydispersed suspensions with partial absorption. *Electromagnetic Scattering*, M. Kerker, Ed., Pergamon, 171–189.

Hansen, J. E., 1971: Multiple scattering of polarized light in planetary atmospheres. Part I. The doubling method. *J. Atmos. Sci.*, **36**, 508–518.

——, and L. D. Travis, 1974: Light scattering in planetary atmospheres. *Space Sci. Rev.*, **16**, 527–610.

Henyey, L. C., and J. L. Greenstein, 1941: Diffuse radiation in the galaxy. *Astrophys. J.*, **93**, 70–83.

Irvine, W. M., 1968: Multiple-scattering by large particles. II. Optically thick layers. *Astrophys. J.*, **152**, 823–834.

Joseph, J. H., W. J. Wiscombe and J. A. Weinman, 1976: The delta-Eddington approximation for radiative flux transfer. *J. Atmos. Sci.*, **33**, 2452–2459.

Kawata, Y., and W. M. Irvine, 1970: The Eddington approximation for planetary atmospheres. *Astrophys. J.*, **160**, 787–790.

King, L. V., 1913: On the scattering and absorption of light in gaseous media, with applications to the intensity of sky radiation. *Phil. Trans. Roy. Soc. London*, **A212**, 375–433.

King, M. D., 1981: A method for determining the single scattering albedo of clouds through observation of the internal scattered radiation field. *J. Atmos. Sci.*, **38**, 2031–2044.

——, 1983: Number of terms required in the Fourier expansion of the reflection function for optically thick atmospheres. *J. Quant. Spectrosc. Radiat. Transfer*, **30**, 143–161.

——, and B. M. Herman, 1979: Determination of the ground albedo and the index of absorption of atmospheric particulates by remote sensing. Part I: Theory. *J. Atmos. Sci.*, **36**, 163–173.

——, and Harshvardhan, 1986: Comparative accuracy of the albedo, transmission and absorption for selected radiative transfer approximations. NASA Ref. Pub. 1160, 41 pp.

——, —— and A. Arking, 1984: A model of the radiative properties of the El Chichón stratospheric aerosol layer. *J. Climate Appl. Meteor.*, **23**, 1121–1137.
Liou, K. N., 1973: A numerical experiment on Chandrasekhar's discrete-ordinate method for radiative transfer: Applications to cloudy and hazy atmospheres. *J. Atmos. Sci.*, **30**, 1303–1326.
——, 1974: Analytic two-stream and four-stream solutions for radiative transfer. *J. Atmos. Sci.*, **31**, 1473–1475.
Meador, W. E., and W. R. Weaver, 1980: Two-stream approximations to radiative transfer in planetary atmospheres: A unified description of existing methods and a new improvement. *J. Atmos. Sci.*, **37**, 630–643.
Paige, D. A., and A. P. Ingersoll, 1985: Annual heat balance of martian polar caps: Viking observations. *Science*, **228**, 1160–1168.
Schaller, E., 1979: A delta-two-stream approximation in radiative flux calculations. *Contrib. Atmos. Phys.*, **52**, 17–26.
Shettle, E. P., and J. A. Weinman, 1970: The transfer of solar irradiance through inhomogeneous turbid atmospheres evaluated by Eddington's approximation. *J. Atmos. Sci.*, **27**, 1048–1055.
Sobolev, V. V., 1975: *Light Scattering in Planetary Atmospheres* (Transl. by W. M. Irvine). Pergamon, 256 pp.
Stephens, G. L., 1984: The parameterization of radiation for numerical weather prediction and climate models. *Mon. Wea. Rev.*, **112**, 826–867.
van de Hulst, H. C., 1968a: Radiative transfer in thick atmospheres with an arbitrary scattering function. *Bull. Astron. Inst. Netherlands*, **20**, 77–86.
——, 1968b: Asymptotic fitting, a method for solving anisotropic transfer problems in thick layers. *J. Comput. Phys.*, **3**, 291–306.
——, 1980: *Multiple Light Scattering. Tables, Formulas, and Applications*, Vols. 1 and 2. Academic Press, 739 pp.
Welch, R. M., and W. G. Zdunkowski, 1982: Backscattering approximations and their influence on Eddington-type solar flux calculations. *Contrib. Atmos. Phys.*, **55**, 28–42.
Wiscombe, W. J., and G. W. Grams, 1976: The backscattered fraction in two-stream approximations. *J. Atmos. Sci.*, **33**, 2440–2451.
——, and J. H. Joseph, 1977: The range of validity of the Eddington approximation. *Icarus*, **32**, 362–377.
——, and S. G. Warren, 1980: A model of the spectral albedo of snow. I: Pure snow. *J. Atmos. Sci.*, **37**, 2712–2733.
Zdunkowski, W. G., R. M. Welch and G. Korb, 1980: An investigation of the structure of typical two-stream-methods for the calculation of solar fluxes and heating rates in clouds. *Contrib. Atmos. Phys.*, **53**, 147–166.